CALCULUS II
WHAT IS e?

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ABSTRACT. Let a be a positive integer, and let x be a real number. We wish to consider the question, what does a^x mean? Using properties of exponentials we easily obtain from the case when x is a positive integer, we derive the best meaning for the cases when $x = 0$, x is a negative integer, and x is a rational number.

From here, we need to address the case where x is an irrational number; to do this, we describe how irrational numbers are the limit of a bounded increasing sequence of rational numbers.

Next, we derive the famous constant e as the limit of a bounded increasing sequence motivated by consideration of the exponential function obtained from the example of compound interest.

1. BOUNDED INCREASING SEQUENCES

1.1. **Lists of Numbers.** Consider the following lists of numbers:

(a) $1, 2, 3, 4, 5 \ldots$
(b) $1, 4, 9, 16, 25 \ldots$
(c) $1, 3, 5, 7, 9 \ldots$
(d) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$
(e) $1, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$
(f) $3, 3.1, 3.14, 3.141, 3.1415, \ldots$

Dot dot dot is intended to indicate that the list continues according to a pattern which one can easily decern. We believe that we understand what the next number in each list should be:

$$\textbf{(a)}\, 6, \quad \textbf{(b)}\, 36, \quad \textbf{(c)}\, 11, \quad \textbf{(d)}\, \frac{1}{6}, \quad \textbf{(e)}\, \frac{5}{6}, \quad \textbf{(f)}\, 3.14159.$$

This is because we understand the lists in terms of what they are; we can describe the lists in words:

(a) positive integers
(b) squares of positive integers
(c) odd positive integers
(d) reciprocals of positive integers
(e) quotients of consecutive integers
(f) decimal estimates of π

Since each list inspires a well-defined way of describing the n^{th} value, we can think of each list as a function which takes positive integers and produces values.

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In fact, we believe that we can say what the generic n^{th} element of the list should be, by giving a formula for it, in terms of n:

$$\textbf{(a)}\, n, \quad \textbf{(b)}\, n^2, \quad \textbf{(c)}\, 2n-1, \quad \textbf{(d)}\, \frac{1}{n}, \quad \textbf{(e)}\, \frac{n-1}{n}, \quad \textbf{(f)}\, \frac{\lfloor 10^{n-1}\pi \rfloor}{10^{n-1}}.$$

1.2. **Sequences.** The lists we have discussed are examples of a certain type of function known as a sequence.

Definition 1. A *sequence* is a function whose domain is the set of positive integers.

For example,

$$a(n) = n^2$$

defines a sequence, where n ranges over the positive integers. It is standard to write a_n to mean $a(n)$, so in this case,

$$a_1 = 1,\ a_2 = 4,\ a_3 = 9,\ a_4 = 16,\ a_5 = 25,\ \ldots$$

and so forth. In this notation, it is common to write (a_n) to indicate the entire function.

We can graph this sequence; it is the parabolic graph of $f(x) = x^2$, with a solid dot above every positive integer.

A sequence may be indicated by a list, where the pattern is clear. For example, the list for the sequence (a_n) given by $a_n = n^2$ is

$$1, 4, 9, 16, 25, 36, \ldots$$

Consider the sequence (a_n) given by $a_n = \frac{1}{n}$. As n gets bigger and bigger, $\frac{1}{n}$ gets smaller and smaller, and in fact, gets closer and closer to 0. By selecting a large enough n, we have $\frac{1}{n}$ as close to 0 as we wish. We say that 0 is the *limit* of the sequence as n goes to infinity, and write

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

Next consider the sequence (a_n) given by $a_n = \frac{n-1}{n} = 1 - \frac{1}{n}$. Since $\frac{1}{n}$ gets arbitrarily close to zero as n increases, $1 - \frac{1}{n}$ gets arbitrarily close to 1, so

$$\lim_{n\to\infty} \frac{n-1}{n} = 1.$$

1.3. **Increasing Sequences.** The squares, odd numbers, and primes get larger and larger as we proceed down their lists, as do reciprocals of consecutive integers.

Definition 2. Let (a_n) be a sequence. We say that (a_n) is *increasing* if $a_n < a_{n+1}$ for every positive integer n.

The sequences of squares, odd numbers, and prime numbers are examples of increasing sequences. These sequence grow without bound, which is to say, they get arbitrarily large.

The sequence given by $a_n = 1 - \frac{1}{n}$ is increasing; however, it does not get arbitrarily large, and in fact is never bigger than 1.

Visualize this by considering the rational function $f(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$. This function has an x-intercept at the point $(1, 0)$, a vertical asymptote at $x = 0$, and a horizontal asymptote at $y = 1$. For positive x, the graph lies below the line $y = 1$. We use the graph of this rational function to produce the graph of $a_n = 1 - \frac{1}{n}$ as solid dots above integers residing on the graph of f.

1.4. **Bounded Increasing Sequences.** If an increasing sequence every gets larger than a number L, it can never go back down and get arbitrarily close to L. Thus, if an increasing sequence has a limit L, we see that it never gets bigger than L; we say that the sequence is bounded by L.

Definition 3. Let (a_n) be an increasing sequence, and let M be a real number. We say that (a_n) is *bounded* if there exists a real number $M > 0$ such that $a_n < M$ for all n. We call M an *upper bound*.

The line $y = M$ is like a "ceiling", above which the graph of the sequence never rises. If we imagine the ceiling getting lower and lower until it cannot be lowered any more without crossing the graph of the sequence, we see that there exists an upper bound L that is less than all other upper bounds. This is called the *least upper bound*. The following theorem states what we intuitively see to be the case.

Theorem 1. *A bounded increasing sequence (a_n) has a least upper bound L, and*

$$\lim_{n\to\infty} a_n = L.$$

For example, $1 - \frac{1}{n}$ has 3, 2, and 1.5 as upper bounds, but 1 is its least upper bound.

1.5. **Sequences of Rational Numbers.** Each real number is the limit of an increasing sequence of rational numbers, as we show by example.

Let x be the unique positive real number whose square is two; we write $x = \sqrt{2}$. It can easily be shown that x is irrational.

Proposition 1. *$\sqrt{2}$ is irrational.*

Proof. Let us suppose that $\sqrt{2}$ is rational, and find why this cannot be true by producing a contradiction.

Assume that $\sqrt{2}$ is rational. Then we may write $\sqrt{2}$ as a quotient of integers which have no common prime factors, so there exist integers a and b such that $\sqrt{2} = \frac{a}{b}$, and at least one of them is odd.

Squaring gives $2 = \frac{a^2}{b^2}$, so $2b^2 = a^2$. Thus a^2 is even, which implies that a is even. Thus $a = 2c$ for some integer c, so $2b^2 = (2c)^2 = 4c^2$, which implies that $b^2 = 2c^2$. This implies that b^2 is even, whence b is even, a contradiction. □

Since $\sqrt{2}$ is real, it has a decimal expansion:

$$\sqrt{2} = 1.4142\ldots.$$

The decimal expansion for $\sqrt{2}$ produces a sequence of rational numbers:

$$1,\ 1.4 = \frac{14}{10},\ 1.41 = \frac{141}{100},\ 1.414 = \frac{1414}{1000},\ 1.4142 = \frac{14142}{10000}, ...$$

This entire sequence is obviously bounded above by 2; thus it has a least upper bound. That least upper bound is in fact $\sqrt{2}$, and this sequence converges to $\sqrt{2}$.

In general, if x is an irrational number, and (x_n) is a sequence defined by

$$x_n = \frac{\lfloor 10^{n-1}x \rfloor}{10^{n-1}}.$$

Then (x_n) is a bounded increasing sequence of rational numbers whose limit is x.

2. Exponents

Let a be a positive real number, and let x be a real number. We ask, what is the meaning of a^x?

2.1. When x is a positive integer.

Let $n = x$, and assume that n is a positive integer. Then a^n is defined to mean the product of n numbers whose value is a:

$$a^n = \underbrace{a \times \cdots \times a}_{n \text{ times}}.$$

From this, we obtain two significant properties.

(E1) $a^{m+n} = a^m \cdot a^n$

(E2) $(a^m)^n = a^{mn}$

To see this, write

$$a^{m+n} = \underbrace{a \times \cdots \times a}_{m+n \text{ times}} = \underbrace{a \times \cdots \times a}_{m \text{ times}} \times \underbrace{a \times \cdots \times a}_{n \text{ times}} = a^m \times a^n.$$

and

$$(a^m)^n = (\underbrace{a \times \cdots \times a}_{m \text{ times}})^n = \underbrace{(\underbrace{a \times \cdots \times a}_{m \text{ times}}) \times \cdots \times (\underbrace{a \times \cdots \times a}_{m \text{ times}})}_{n \text{ times}} = \underbrace{a \times \cdots \times a}_{mn \text{ times}} = a^{mn}.$$

We wish to extend the meaning of a^x so that it is defined for any real number x, in such a way that the properties **(E1)** and **(E2)** remain true.

2.2. When $x = 0$.

Consider the case when $x = 0$. We multiply a times a^0; whatever a^0 means, if property **(E1)** is to remain true, we have

$$aa^0 = a^1 a^0 = a^{1+0} = a^1 = a.$$

Dividing both sides by a gives

$$a^0 = 1.$$

2.3. When x is a negative integer.

Consider the case when x is a negative integer, so that $x = -n$ for some positive integer n. For **(E1)** to remain true, we must have

$$a^n a^x = a^{n+x} = a^0 = 1.$$

In this case,

$$a^{-n} = \frac{1}{a^n}.$$

2.4. When x is rational.

Consider the case when $x = \frac{1}{n}$, where n is a positive integer. For **(E2)** to remain true, we must have

$$(a^{1/n})^n = a^{n/n} = a^1 = a.$$

Thus, $a^{1/n}$ is the unique number whose n^{th} power is a; that is,

$$a^{1/n} = \sqrt[n]{a}.$$

Consider the case when $x = \frac{m}{n}$, where m and n are positive integers. Then **(E2)** produces $a^{m/n} = (a^m)^{1/n}$, so

$$a^{m/n} = \sqrt[n]{a^m}.$$

2.5. **When x is irrational.** We now consider the case when x is irrational. This is the hardest step.

Integers are obtained from natural numbers by algebraic considerations (defining subtraction), and rational numbers are obtained from integers by additional algebraic considerations (defining division); however, real numbers are obtained from rationals by geometric considerations (filling in gaps in the number line).

There is an additional property of exponents which is important in this context:

(E3) if $1 < a$ and $r < s$, then $a^r < a^s$

This is true when x is any rational number, and we wish it to remain true for any real number.

We line up all of the rationals by the order relation $<$, and see that there are gaps in the line; so, too, we can line up all of the numbers of the form a^q where q is rational, and see that there are gaps in the line; we hope to fill these gaps by numbers of the form a^x, where x is irrational.

Let x be an irrational number, and define the sequence (x_n) by

$$x_n = \frac{\lfloor 10^{n-1}x \rfloor}{10^{n-1}},$$

so that (x_n) is a bounded increasing sequence of rational numbers, and

$$x = \lim_{n\to\infty} x_n.$$

This is a sequence of decimal estimates of x of increasing accuracy; it is an increasing sequence that converges to x.

Since x_n is rational, a^{x_n} is defined. Consider the sequence (a^{x_n}); by Property **(E3)**, this is an increasing sequence of real numbers which is bounded above by $a^{\lceil x \rceil}$. Thus, it converges. We define

$$a^x = \lim_{n\to\infty} a^{x_n}.$$

This definition extends the previous definitions in such a way as to preserve properties **(E1)**, **(E2)**, and **(E3)**.

2.6. **Exponential Functions.** Let a be a positive real number. Now that a^x is defined for any real number x, we see that, by letting x vary throughout the real numbers, we obtain a function.

The *base a exponential function* is the function a^x. This function has these properties:

(a) $a^0 = 1$
(b) $a^1 = a$
(c) $a^{r+s} = a^r a^s$
(d) $(a^r)^s = a^{rs}$
(e) $r < s \Rightarrow a^r < a^s$, if $a > 1$
(f) $r < s \Rightarrow a^r > a^s$, if $0 < a < 1$

Our first examples of exponential functions will be those which compute compound interest. From this, we derive the transcendental number e.

3. The Number e

3.1. Periodic Compound Interest. Suppose we invest 1000 dollars at an interest rate of 10 percent compounded annually. The amount we have invested remains the same until one year passes, at which point 10 percent of the amount is added to the total. If we let A_t denote the amount invested after t years, then

- $A_0 = 1000$
- $A_1 = 1000 + (0.1)1000 = 1100$
- $A_2 = 1100 + (0.1)1100 = 1210$
- $A_3 = 1210 + (0.1)1210 = 1331$

We see that the rate at which this grows increases year by year; but the pattern is obscure. It is actually easier to see the pattern if we think more generally.

Let r be the annual interest rate, A_0 the initial investment, and A_t the amount after t years. Then

- $A_1 = A_0 + rA_0 = A_0(1 + r)$
- $A_2 = A_1 + rA_1 = A_1(1 + r) = A_0(1 + r)^2$
- $A_3 = A_2 + rA_2 = A_2(1 + r) = A_0(1 + r)^3$
- $A_t = A_0(1 + r)^t$

Suppose that, instead of compounding annually, we compound quarterly; that is, every three months, or four times per year. Then, the periodic interest rate is the annual rate divided by four.

- $A_{1/4} = A_0 + (\frac{r}{4})A_0 = A_0(1 + \frac{r}{4})$
- $A_{1/2} = A_{1/4} + (\frac{r}{4})A_{1/4} = A_{1/4}(1 + \frac{r}{4}) = A_0(1 + \frac{r}{4})^2$
- $A_1 = A_0(1 + \frac{r}{4})^4$
- $A_t = A_0(1 + \frac{r}{4})^{4t}$

Generalize this further; let k denote the number of periods per year, so that we compound k times per year. Then, there are k times every year when we the amount in the account by $(1 + \frac{r}{k})$; these gives

$$A_t = A_0\left(1 + \frac{r}{k}\right)^{kt},$$

where r is the annual rate, k is the number of periods per year, and A_t is the amount after t years.

The more periods per year, the faster the amount grows, as this table demonstrates. We let the annual rate r be ten percent and the initial investment A_0 be one thousand. We compute the amount after five years for various values of k, to the nearest dollar:

k	A_0	A_1	A_2	A_3	A_4	A_5
1	1000	1100	1210	1331	1464	1611
2	1000	1103	1216	1340	1477	1629
4	1000	1104	1218	1345	1485	1639
12	1000	1105	1220	1348	1489	1645
365	1000	1105	1221	1350	1492	1649
8760	1000	1105	1221	1350	1492	1649

This table demonstrates two facts:

- as k increases, the investment grows faster;
- as k increases, the rate at which the investment grows faster slows down.

3.2. **Continuous Compound Interest.** We wish to define continuously compounded interest as the limit of periodically compounded interest as the k goes to infinity. Thus we fix A_0, r, and t, and attempt to understand the expression

$$\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt}.$$

To do this, we define a new variable n by $n = \frac{k}{r}$, so that $k = nr$ and $\frac{r}{k} = \frac{1}{n}$. Since r is fixed, n goes to infinity as k goes to infinity. We compute

$$\begin{aligned}\lim_{k\to\infty} A_0\left(1+\frac{r}{k}\right)^{kt} &= \lim_{n\to\infty} A_0\left(1+\frac{1}{n}\right)^{nrt}\\ &= \lim_{n\to\infty} A_0\left[\left(1+\frac{1}{n}\right)^{n}\right]^{rt}\\ &= A_0\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{rt}.\end{aligned}$$

This computation tells us that continuously compounded interest may be computed using an exponential function whose base is the limit of the sequence $(1+\frac{1}{n})^n$; it can be show that this is an increasing sequence which is bounded above by 3, so it converges. The number it converges to turns out to be so important in mathematics that we give it a special name.

Define

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}.$$

Then, the equation which computes the amount A_t for continuously compounded interest is

$$A_t = A_0 e^{rt}.$$

We estimate e by computing a few values:

n	$(1+\frac{1}{n})^n$	estimate
1	$(2)^1$	2.000000
2	$(1.5)^2$	2.250000
4	$(1.25)^4$	2.441406
10	$(1.1)^{10}$	2.593742
100	$(1.01)^{100}$	2.704813
1000	$(1.001)^{1000}$	2.716923
10000	$(1.0001)^{10000}$	2.718145
100000	$(1.00001)^{100000}$	2.718268
∞	e	2.718281

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